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Applications of Approximation Theory to the Initial Value Problem

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1. INTRODUCTION

In recent years, several papers have been devoted to best approximating in some sense the solutions to various types of nonlinear differential and integro-differential equations (see, for example, [1, 4, 7–9]). These papers are generally concerned with proving that a best approximation exists and with showing that an appropriate sequence of best approximations converges to a solution of the differential or integro-differential equation. In the previously mentioned references, the best approximation problem is ordinarily nonlinear and consequently computational techniques are not readily available.

In this paper we propose to reexamine the best approximation problem as posed in [1, 4, 7–9] and discuss alternatives that may result in computationally obtainable best approximations. Although our discussion is limited to the initial value problem IVP

$$\begin{aligned} \ddot{x}(t) &= f(t, x(t), \dot{x}(t)), \\ x(0) = c_0, &\qquad \dot{x}(0) = c_1, \end{aligned} \tag{1.1}$$

on the interval $I_\alpha = [-\alpha, \alpha]$, the concepts discussed are adaptable to higher-order differential equations and integrodifferential equations.

2. The Best Approximation Problem

Let

$$\mathbf{P}_k = \{p(A, t): p(A, t) = c_0 + c_1 t + a_2 t^2 + \cdots + a_k t^k\}, \tag{2.1}$$

where the vector $A = (a_2, ..., a_k)$ is an element of E_{k-1}, the $(k-1)$-dimensional Euclidean space. The best approximation problem (essentially that of any of the above references) is to find an element $p(A_k^*, t) \in \mathbf{P}_k$ such that

$$\inf_{p \in \mathbf{P}_k} \sup_{I_\alpha} | \ddot{p}(A, t) - f(t, p(A, t), \dot{p}(A, t))| = \sup_{I_\alpha} | \ddot{p}(A_k^*, t) - f(t, p(A_k^*, t), \dot{p}(A_k^*, t))|. \tag{2.2}$$

We designate establishing the existence of a $p(A_k^*, t) \in \mathbf{P}_k$ that satisfies (2.2) to be the *best simultaneous approximation problem*. If such a $p(A_k^*, t)$ does exist, then this polynomial is the *best simultaneous approximate solution* (BAS) of degree k to the IVP (1.1) on the interval $[-\alpha, \alpha]$.

If $y(t)$ is the solution to (1.1) and if $\{p(A_k^*, t)\}_{k=2}^{\infty}$ is a sequence of best approximate solutions, one for each k, then a fundamental question is whether or not

$$\lim_{k\to\infty} \| p(A_k^*, \cdot) - y \|_\alpha = 0. \tag{2.3}$$

Hereafter,

$$\| h \|_\alpha = \sup_{[-\alpha, \alpha]} | h(t)|. \tag{2.4}$$

References [1, 4, 7–9] all consider this basic question. It should be noted that in some of these papers, norms other than the Chebyshev norm are considered. Our attention is restricted to the Chebyshev norm (2.4).

A basic difficulty of the type of approximation described in (2.2) is that if $f(t, x, \bar{x})$ is nonlinear in x or $\bar{x}$, then (2.2) is a nonlinear approximation problem; therefore the BAS of degree k is frequently not easily computable, (see [6]).

Another way of viewing the approximation problem (2.2) is as follows. Again let $A = (a_2, ..., a_k) \in E_{k-1}$, and let $p(A, t) = c_0 + c_1 t + a_2 t^2 + \cdots + a_k t^k$ be the element of $\mathbf{P}_k$ corresponding to the vector A. Then (2.2) may be viewed as finding the $A_k^* \in E_{k-1}$ (if it exists) that minimizes $\| G(A, \cdot)\|_\alpha$, where

$$G(A, t) = \ddot{p}(A, t) - f(t, p(A, t), \dot{p}(A, t)). \tag{2.5}$$

If $\{G(A, t)\}_{A \in E_{k-1}}$ is a varisolvent family in the sense of Rice [11], then classical approximation theory techniques could be employed despite the fact that $G(A, t)$ may be nonlinear in the parameters $(a_2, ..., a_k)$. We shall demonstrate shortly, however, that simple nonlinearities in $f(t, x, \dot{x})$ destroy all possibilities of varisolvence. Thus, little is known as far as computing a best approximation for fixed k in the sense of (2.2).

In 1969, Olson [10] and Weinstein introduced a possible alternative to the approximation problem (2.2). Although their work is basically for the first-order counterpart to (1.1) (some of their examples are for second-order equations), we will discuss the procedure in terms of (1.1). Let $p(A_{20}, t) = c_0 + c_1 t$. Here A_{20} is the zero vector in E_1. The first subscript is one more than the dimension of A_{20} and corresponds to the class of polynomials $\mathbf{P}_2$, and the second subscript denotes the particular iterate in the algorithm presently being described.

Solve the linear best approximation problem

$$\inf_{A\in E_1} \sup_{I_\alpha} | \ddot{p}(A, t) - f(t, p(A_{20}, t), \dot{p}(A_{20}, t))|$$

via the second algorithm of Remes. Let $\ddot{p}(A_{21}, t)$ be this best approximation, and set

$$p(A_{21}, t) = c_0 + c_1 t + \int_0^t (t - s)\, \ddot{p}(A_{21}, s)\, ds.$$

Then $p(A_{21}, t) \in \mathbf{P}_2$. Now solve the linear approximation problem

$$\inf_{A\in E_1} \sup_{I_\alpha} | \ddot{p}(A, t) - f(t, p(A_{21}, t), \dot{p}(A_{21}, t))|.$$

Designate by $\ddot{p}(A_{22}, t)$ this best approximation and let

$$p(A_{22}, t) = c_0 + c_1 t + \int_0^t (t - s)\, \ddot{p}(A_{22}, s)\, ds.$$

Again, $p(A_{22}, t) \in \mathbf{P}_2$. Continuing this process results in a sequence $\{p(A_{2n}, t)\}_{n=0}^{\infty} \subseteq \mathbf{P}_2$. If a subsequence converges to a $p(\bar{A}_2, t) \in \mathbf{P}_2$, define $p(A_{30}, t) = p(\bar{A}_2, t)$, where now $A_{30} \in E_2$ and $p(A_{30}, t) \in \mathbf{P}_3$. That is, if $\bar{A}_2 = (\bar{a}_{22})$, then $A_{30} = (\bar{a}_{22}, 0) \in E_2$. The algorithm continues in E_2 by now solving the approximation problem

$$\inf_{A\in E_2} \sup_{I_\alpha} | \ddot{p}(A, t) - f(t, p(A_{30}, t), \dot{p}(A_{30}, t))|.$$

Let $\ddot{p}(A_{31}, t)$ be this best approximation, and again set

$$p(A_{31}, t) = c_0 + c_1 t + \int_0^t (t - s)\, \ddot{p}(A_{31}, s)\, ds.$$

Then $p(A_{31}, t) \in \mathbf{P}_3$. Continuing, one obtains a sequence $\{p(A_{3n}, t)\}_{n=0}^{\infty} \subseteq \mathbf{P}_3$.

If a subsequence converges to $p(\bar{A}_3, t) \in \mathbf{P}_3$, define $p(A_{40}, t) = p(\bar{A}_3, t)$, where if $\bar{A}_3 = (\bar{a}_{23}, \bar{a}_{33})$, then $A_{40} = (\bar{a}_{23}, \bar{a}_{33}, 0)$.

Continuing this process (assuming convergence) results in a sequence of polynomials $\{p(\bar{A}_k, t)\}_{k=2}^{\infty}$ with $p(\bar{A}_k, t) \in \mathbf{P}_k$.

Obvious questions of interest include: (a) When does the sequence $\{p(A_{kn}, t)\}_{n=0}^{\infty}$ possess a subsequence that converges uniformly to an element $p(\bar{A}_k, t)$ of $\mathbf{P}_k$? (b) How does $p(\bar{A}_k, t)$ compare to $p(A_k^*, t)$, the BAS of degree k? (c) Does a subsequence of $\{p(\bar{A}_k, t)\}_{k=2}^{\infty}$ converge uniformly to the solution $y(t)$ of (1.1), even though the sequences $\{p(\bar{A}_k, t)\}_{k=2}^{\infty}$ and $\{p(A_k^*, t)\}_{k=2}^{\infty}$ may differ? (d) How feasible computationally is the algorithm? It should be noted in regard to question (d) that one would ordinarily start the algorithm with k fairly large initially.

Parts of these questions are considered in [10] but for each fixed k these results depend on the $G(A, t)$ of (2.5) satisfying property Z of degree $k - 1$ at A^* (see Rice [11, p. 3]); that is, if $A^* \in E_{k-1}$ and if A is any other element of E_{k-1}, then $G(A^*, t) - G(A, t)$ can have at most $k - 2$ zeros on $[-\alpha, \alpha]$. (Again, $p(A^*, t)$ and $p(A, t)$ are elements of $\mathbf{P}_k$). It is not likely that property Z will be satisfied, even if $f(t, x, \dot{x})$ is only "mildly" nonlinear

EXAMPLE 1.

$$\ddot{x}(t) - x^2(t) = 1, \qquad I_\alpha = [-\alpha, \alpha],$$
$$x(0) = \dot{x}(0) = 0.$$

Let the approximation be from $\mathbf{P}_2$. Thus every element is of the form $p(A, t) = at^2$, and $A = (a)$. If $A^* = (a^*)$ is given then

$$G(A^*, t) - G(A, t) = (a^* - a)[2 - (a^* + a)\, t^4].$$

But then $G(A^*, t) - G(A, t) = 0$ for $a \neq a^*$ providing $a = (2/t^4) - a^*$. Thus given an a^*, there is a $\bar{t} \in [-\alpha, \alpha]$ and a corresponding $\bar{a} \neq a^*$ such that $\bar{a} = (2/(\bar{t})^4) - a^*$. That is, there is a $\bar{t} \in [-\alpha, \alpha]$ and an $\bar{A} \neq A^*$ such that $G(A^*, \bar{t}) - G(\bar{A}, \bar{t}) = 0$. Thus at A^* property Z of degree one is violated.

3. BEHAVIOR OF THE ALGORITHM

In this section, we consider an example that demonstrates the types of answers to questions (a)–(d) that can be expected. The notation of section two was employed primarily to identify the types of approximation problems of this paper with the classical approximation theory problem above (2.5).

For simplicity, some notational changes are now made. If $p_k^*(t)$ (instead of the previously designated $p(A_k^*, t)$) satisfies

$$\inf_{p\in\mathbf{P}_k} \sup_{[-\alpha,\alpha]} | \ddot{p}(t) - f(t, p(t), \dot{p}(t))| \\ = \sup_{I_\alpha} | \ddot{p}_k^*(t) - f(t, p_k^*(t), \dot{p}_k^*(t))|, \tag{3.1}$$

then $p_k^*(t)$ is the *best simultaneous approximate solution* to the initial value problem (1.1). Hereafter, $p_k^*(t)$ will be called the BAS of degree k. If $p_{kn}(t) = p(A_{kn}, t)$ of the previous section, then the algorithm for degree k of section two becomes

$$\inf_{p\in\mathbf{P}_k} \sup_{I_\alpha} | \ddot{p}(t) - f(t, p_{k,n-1}(t), \dot{p}_{k,n-1}(t))| \\ = \sup_{I_\alpha} | \ddot{p}_{kn}(t) - f(t, p_{k,n-1}(t), \dot{p}_{k,n-1}(t))|. \tag{3.2}$$

If $p_k(t) = p(\bar{A}_k, t)$ of the previous section (that is, $p_k(t)$ is a cluster point of the sequence $\{p_{kn}(t)\}_{n=0}^{\infty}\}$, then under appropriate conditions (Section 4) $p_k(t)$ is called a *simultaneous approximation substitute* of degree k, SAS.

Thus the possibly nonlinear approximation problem (3.1) is replaced by the linear approximation problem (3.2), and we are interested in the various properties of BAS and SAS.

EXAMPLE 2.

$$\ddot{x}(t) - \tfrac{1}{3}x^2(t) = 0, \qquad I_\alpha = [-\alpha, \alpha], \\ x(0) = 1, \qquad \dot{x}(0) = 0.$$

It can be shown that Example 2 has a unique solution for $\alpha < 2$. The approximation problem (3.1) for $k = 2$ is then to find an a^* that satisfies

$$\inf_{a\in E_1} \sup_{I_\alpha} | 2a - \tfrac{1}{3}(1 + at^2)^2 | = \sup_{I_\alpha} | 2a^* - \tfrac{1}{3}(1 + a^*t^2)^2 |, \tag{3.3}$$

where $\mathbf{P}_2 = \{p(t): p(t) = 1 + at^2\}$.

The algorithm (3.2) for $k = 2$ is to find the a_{n+1} that satisfies

$$\inf_{a\in E_1} \sup_{I_\alpha} | 2a - \tfrac{1}{3}(1 + a_nt^2)^2 | = \sup_{I_\alpha} | 2a_{n+1} - \tfrac{1}{3}(1 + a_nt^2)^2 |. \tag{3.4}$$

That is, the initial guess for the algorithm is $p_{20}(t) = 1$, and then $p_{2n}(t) = 1 + a_nt^2$. Thus for $k = 2$ the algorithm generates the sequence

$$\{p_{2n}(t)\}_{n=0}^{\infty} \tag{3.5}$$

The table below compares the two approximation problems (3.3) and (3.4) for various values α.

α^2	SAS	BAS
1.44	$1 + 0.23148t^2$	$1 + 0.23148t^2$
2.2	$1 + 0.33435t^2$	$1 + 0.16325t^2$
2.7	Does not exist	$1 + 0.04115t^2$

Thus the sequence (3.5) converges to $p_2(t) = 1 + 0.23248t^2$ for $\alpha^2 = 1.44$ and SAS $=$ BAS; for $\alpha^2 = 2.2$ the sequence (3.5) converges to $p_2(t) = 1 + 0.33435t^2$ but SAS does not equal BAS; finally (3.5) does not converge for $\alpha^2 = 2.7$, nor does any subsequence of (3.5) converge, and consequently no SAS exists.

Thus in general, one might expect that the algorithm may not converge on the entire interval of definition of the IVP (1.1), but rather on some smaller interval. Also it might be anticipated that the SAS and BAS of degree k may be equal on a sufficiently small interval.

4. Theory of the Algorithm

In this section we first consider the existence of a SAS of degree k. It is assumed that f is a real-valued function from $I \times R^2$ into R, where $I = [-a, a]$, $\alpha \leqslant a$, and R represents the set of all real numbers.

Since $f \in C[I \times R^2]$, $|f(t, x, \bar{x})| \leqslant M = M(B)$ whenever $(t, x, \bar{x}) \in I \times R^2$ and $\max(|x|, |\bar{x}|) \leqslant B$. For $u \in C^1[I]$, define $\|u(t)\|_1 = \max(|u(t)|, |\dot{u}(t)|)$, and let $p_0(t) = c_0 + c_1 t$, $t \in I$.

H_1. *Let* $B \geqslant 1 + \max_I(\|p_0\|_1)$. *Choose* I_α *to ensure for all* $t \in I_\alpha$ *that*

$$M \max(2\,|\,t\,|, t^2) \leqslant 1.$$

If $I_{\bar{\alpha}}$ is the maximal interval in I satisfying H_1, then for all $\alpha \leqslant \bar{\alpha}$, I_α also satisfies H_1.

Let

$$S_k = \{p \in \mathbf{P}_k : \|p(t) - p_0(t)\|_1 \leqslant M \max(2\,|\,t\,|, t^2),\ t \in I_\alpha\}. \tag{4.1}$$

Then S_k is a compact, convex subset of $\mathbf{P}_k$. We now define an operator T_k on S_k.

For $x \in C^1[I_\alpha]$, let

$$F[x](t) = f(t, x(t), \dot{x}(t)). \tag{4.2}$$

The set of all polynomials of degree at most $k - 2$ is denoted Q_{k-2}. If

$$\inf_{v \in Q_{k-2}} \| v - F[p]\|_\alpha = \| \bar{v} - F[p]\|_\alpha, \tag{4.3}$$

then $q(t) = p_0(t) + \int_0^t (t - s)\, \bar{v}(s)\, ds$ is in $\mathbf{P}_k$. Finally, if $p \in S_k$, define T_k on S_k by $T_k p = q$. In this case, $(d^2/dt^2)[T_k p] = \bar{v}$, the best approximation to $F[p]$ on I_α from Q_{k-2}.

THEOREM 1. *The mapping T_k is a continuous mapping from S_k into S_k.*

Proof. Let $p \in S_k$. Then from the remarks below (4.3), $T_k p = q$ implies

$$\inf_{v \in Q_{k-2}} \| v - F[p]\|_\alpha = \| \ddot{q} - F[p]\|_\alpha, \tag{4.4}$$

where $q \in \mathbf{P}_k$. Let

$$\ddot{q}(t) - F[p](t) = e(t). \tag{4.5}$$

Equalities (4.4) and (4.5) imply

$$\| e \|_\alpha \leqslant \| F[p]\|_\alpha .$$

Since $p \in S_k$, H_1 implies that $\| F[p]\|_\alpha \leqslant M$, and consequently, $\| e \|_\alpha \leqslant M$. From (4.5),

$$\dot{q}(t) - \dot{p}_0(t) = \int_0^t F[p](s)\, ds + \int_0^t e(s)\, ds$$

and

$$q(t) - p_0(t) = \int_0^t (t - s)\, F[p(s)]\, ds + \int_0^t (t - s)\, e(s)\, ds.$$

These two equations imply that

$$\| q(t) - p_0(t)\|_1 \leqslant M \max(2 \mid t \mid, t^2),$$

and consequently $q \in S_k$. Now let $\epsilon > 0$ be given. If $p \in S_k$, the classical Freud Theorem [2, p. 82] implies for any $\bar{p} \in S_k$ that

$$\left\| \frac{d^2}{dt^2} [T_k p] - \frac{d^2}{dt^2} [T_k \bar{p}] \right\|_\alpha \leqslant \lambda_p \| F[p] - F[\bar{p}]\|_\alpha . \tag{4.6}$$

Since f is uniformly continuous on compact subsets of $I \times R^2$, there exists a $\delta > 0$ such that if $\| p - \bar{p} \|_\alpha < \delta$, then

$$\| F[p] - F[\bar{p}]\|_\alpha < \frac{2}{\alpha^2} \frac{\epsilon}{\lambda_p} .$$

Therefore (4.6) yields

$$\left\| \frac{d^2}{dt^2} [T_k p] - \frac{d^2}{d^2 t} [T_k \bar{p}] \right\|_\alpha < \frac{2}{\alpha^2} \epsilon.$$

This implies $\| T_k p - T_k \bar{p} \|_\alpha < \epsilon$, and consequently, T_k is a continuous mapping from S_k into S_k, $k \geqslant 2$.

COROLLARY 1. *The mapping T_k has a fixed point in S_k, $k \geqslant 2$.*

Proof. Since S_k is a compact, convex subset of Q_k the result follows from Theorem 1 and the Schauder fix point theorem [13, p. 25].

We summarize the results of Theorem 1 and Corollary 1. There exists a polynomial $p \in \mathbf{P}_k$ such that if we seek the polynomial $\bar{v} \in Q_{k-2}$ that best approximates the known continuous function $F[p]$ on I_α, then this polynomial is just the second derivative of the original polynomial p. Hereafter, any fixed point of T_k is defined to be a *SAS of degree k* for (1.1) on I_α. We have established for each I_α, $\alpha \leqslant \bar{\alpha}$, a SAS of degree k exists for $k \geqslant 2$.

We now examine conditions that ensure algorithm (3.2) generates a SAS of degree k.

COROLLARY 2. *Let I_α satisfy H_1. Then the sequence $\{p_{kn}\}_{n=0}^{\infty}$ generated by* (3.2) *has a cluster point $p_k \in S_k$, $k \geqslant 2$.*

Proof. Since $p_{k0} \in S_k$ and $T_k p_{kn} = p_{k,n+1}$, the proof of Theorem 1 guarantees that $\{p_{kn}\}_{n=0}^{\infty} \subseteq S_k$. Since S_k is compact, this sequence has a cluster point p_k.

Before proceeding to the next corollary we note since $\{p_{kn}\}_{n=0}^{\infty} \subseteq S_k$ and

$$\| \ddot{p}_{k,n+1} - F[p_{kn}] \|_\alpha \leqslant \| F[p_{kn}] \|_\alpha \leqslant M,$$

where M is independent of k, the sequence $\{\| \ddot{p}_{kn} \|_\alpha\}_{n=0}^{\infty}$, is uniformly bounded in k. Thus Corollary 2 implies the sequence $\{\| \ddot{p}_k \|_\alpha\}_{k=0}^{\infty}$ is bounded.

By the nature of algorithm (3.2), at the $n+1$ step the alternation theorem [2, p. 75] guarantees the existence of an extremal set $X_n = \{t_{1n}, \ldots, t_{kn}\} \subseteq I_\alpha$. The sequence of k-tuples $\{X_n\}_{n=0}^{\infty}$ is contained in the compact set $[I_\alpha]^k$ and consequently has a cluster point $X = \{t_1, \ldots, t_k\}$. Without loss of generality we can assume the subsequences from $\{X_n\}_{n=0}^{\infty}$ and $\{p_{kn}\}_{n=0}^{\infty}$ that converge to X and p_k, respectively, involve the same indices. Consider the error function

$$e_k(t) = \ddot{p}_k(t) - F[p_k](t), \qquad t \in I_\alpha. \tag{4.7}$$

H_2. *Suppose for t_i, $t_{i+1} \in X$ that*

$$e_k(t_i) = -e_k(t_{i+1}), \qquad i = 1, \ldots, k-1.$$

Remark. The point set X and p_k are computable for each k. Hence H_2 is checkable for each k. In particular, if the sequence generated by algorithm (3.2) converges, H_2 is satisfied.

COROLLARY 3. *For each fixed $k \geqslant 2$ let p_k be a cluster point of the sequence generated by algorithm* (3.2) *on I_α, where I_α satisfies H_1. If H_2 is satisfied by the error function* (4.7), *then*

$$\inf_{v \in Q_{k-2}} \| v - F[p_k] \|_\alpha = \| \ddot{p}_k - F[p_k] \|_\alpha .$$

Proof. Let $\{p_{kn(j)}\}_{j=1}^{\infty}$ converge to p_k. Define $e_{kn(j)}$ by

$$\ddot{p}_{k,n(j)+1} - F[p_{kn(j)}] = e_{kn(j)} ,$$

where $(d^2/dt^2)[T_k p_{kn(j)}] = \ddot{p}_{k,n(j)+1}$. Then the remarks below (4.6) and equicontinuity of the sequence $\{F[p_{kn(j)}]\}_{j=1}^{\infty}$ imply that if

$$(d^2/dt^2)[T_k p_k] - F[p_k] = \bar{e}_k ,$$

then $\bar{e}_k(t_i) = -\bar{e}_k(t_{i+1}) = \pm \| \bar{e}_k \|_\alpha$, $i = 1,..., k-1$. Thus,

$$(d^2/dt^2)[T_k p_k](t_i) - F[p_k](t_i) = \bar{e}_k(t_i),$$

and by (4.7)

$$\ddot{p}_k(t_i) - F[p_k](t_i) = e_k(t_i), \qquad i = 1,..., k.$$

Consequently,

$$(d^2/dt^2)[T_k p_k](t_i) - \ddot{p}_k(t_i) = \bar{e}_k(t_i) - e_k(t_i), \tag{4.8}$$

$i = 1,..., k$. But H_2 and the theorem of de La Vallée Poussin [2, p. 77] then imply that $\| \bar{e}_k \|_\alpha \geqslant | e_k(t_i)|$, $i = 1,..., k$. Consequently, (4.8) implies

$$(d^2/dt^2)[Tp_k](t) \equiv \ddot{p}_k(t), \qquad t \in I_\alpha .$$

This corollary implies a cluster point of the sequence generated by algorithm (3.2) is a SAS in the sense of Corollary 1. In the remainder of the paper we assume for each k that the cluster points of algorithm (3.2) are SAS's of degree k on I_α.

The results above basically answer question (a) of Section 2. The next theorem relates directly to question (c). First, a lemma involving fundamental concepts of approximation theory is proven.

LEMMA 1. *The sequence of error functions $\{e_k(t)\}_{k=2}^{\infty}$ defined by* (4.7) *converges uniformly to zero on I_α, $\alpha \leqslant \bar{\alpha}$.*

Proof. Using the notation of Jackson's theorem [12, p. 22], the conclusion of Corollary 3 implies that $\| e_k \|_\alpha = E_{k-2}\{F[p_k]\}$. Thus we can infer from Jackson's theorem that

$$\| e_k \|_\alpha \leqslant 6\omega_k(\alpha/(k-2)), \tag{4.9}$$

where ω_k is the modulus of continuity of $F[p_k](t)$ on I_α (see [12, p. 14, 22]). Let $\epsilon > 0$ be given. Since by Corollary 2 the sequence $\{\ddot{p}_k(t)\}_{k=2}^\infty$ is uniformly bounded, the sequences $\{p_k^{(i)}(t)\}_{k=2}^\infty$, $i = 0, 1$ are uniformly bounded and equicontinuous. Thus the uniform continuity of $f(t, x, \bar{x})$ on any compact set $I_\alpha \times [-N, N]^2$ implies that the sequence $\{F[p_k](t)\}_{k=2}^\infty$ is an equicontinuous family. Consequently there exists a $\delta > 0$ such that $|t - s| \leqslant \delta$ implies that

$$| F[p_k](t) - F[p_k](s)| \leqslant \epsilon,$$

independent of k. Therefore $\omega_k(\delta) \leqslant \epsilon$, independent of k. If K is large enough to ensure that for $k \geqslant K$, $\alpha/(k-2) \leqslant \delta$, then because of the monotonicity of ω_k,

$$\omega_k(\alpha/(k-2)) \leqslant \omega_k(\delta) \leqslant \epsilon.$$

Thus for all $k \geqslant K$,

$$\omega_k(\alpha/(k-2)) \leqslant \epsilon.$$

Inequality (4.9) then implies for all $k \geqslant K$, $\| e_k \|_\alpha \leqslant 6\epsilon$. Consequently $\lim_{k\to\infty} \| e_k \|_\alpha = 0$.

As previously mentioned, the next theorem of this section basically answers question (c) of Section 2. It is of interest to note that this theorem is also an existence theorem for the IVP (1.1), proven via techniques of approximation theory.

THEOREM 2. *Let the f of* IVP (1.1) *be as described above H_1, and suppose $\alpha \leqslant \bar{\alpha}$. Then there is a function $y \in C^2[I]$ and a subsequence $\{p_{k(j)}\}_{j=1}^\infty$ of the sequence $\{p_k\}_{k=2}^\infty$ of* SAS's *satisfying*

$$\lim_{j\to\infty} \| p_{k(j)}^{(i)} - y^{(i)} \|_\alpha = 0, \qquad i = 0, 1, 2.$$

Moreover, y is a solution to the IVP (1.1) *on I_α.*

Proof. Again the sequences $\{p_k^{(i)}(t)\}_{k=2}^\infty$, $i = 0, 1$ are equicontinuous and uniformly bounded on I_α. The Ascoli theorem implies there exists subsequences $\{p_{k(j)}^{(i)}\}_{j=1}^\infty$ such that these sequences converge uniformly on I_α to $y^{(i)}(t)$, $i = 0, 1$, respectively. Then (4.7) implies that

$$\ddot{p}_{k(j)} = e_{k(j)} + F[p_{k(j)}].$$

An application of Lemma 1 establishes that

$$\lim_{j\to\infty} \ddot{p}_{k(j)} = F[y](t). \tag{4.10}$$

But this implies that $\ddot{y}$ exists and

$$\lim_{j\to\infty} \| \ddot{p}_{k(j)} - \ddot{y} \|_\alpha = 0,$$

and consequently (4.10) implies that

$$\ddot{y}(t) = f(t, y(t), \dot{y}(t)).$$

Since each $p_{k(j)} \in \mathbf{P}_{k(j)}$, $y(0) = c_0$ and $\dot{y}(0) = c_1$, and the proof is complete.

In this section questions (a) and (c) were considered in detail. The next section is devoted to at least partially answering the somewhat more difficult question (b).

5. A Comparison of BAS and SAS

The proof of the main theorem of this section establishes for some functions f and for $\alpha \leqslant \bar{\alpha}$ and sufficiently small, the BAS and SAS of degree k are each unique and they are equal. For this proof the function f of (1.1) is restricted as follows.

Condition Q. (i)

$$f(t, x, \bar{x}) = \sum_{i=1}^{n_0} a_i(t)\, x^i + \sum_{i=1}^{n_1} b_i(t)(\bar{x})^i$$
$$+ \sum_{i=1}^{n_3} \sum_{j=1}^{n_2} c_{ij}(t)\, x^i(\bar{x})^j + h(t),$$

where a_i, b_j, c_{ij}, and h are polynomials.

(ii) If p_k is a SAS of degree k, then $F[p_k] \notin Q_{k-2}$.

If (i) is satisfied, then reference [4] guarantees that for each k a BAS exists on I_α. Although condition (i) is restrictive, condition (ii) is generally not a significant additional restriction, since evaluating the f in (i) at $(t, p_k(t), \dot{p}_k(t))$, $t \in I_\alpha$, would ordinarily result in a polynomial of degree strictly greater than k. Condition (ii) is needed in Lemma 4 below.

The first theorem of this section is preceded by two lemmas involving classical ideas of approximation theory.

LEMMA 2. *If $p(t)$ is any polynomial of degree at most n, then*

$$\| p^{(m+1)} \|_\alpha \leqslant \frac{n^2(n-1)^2 \cdots (n-m)^2}{\alpha^{m+1}} \| p \|_\alpha . \tag{5.1}$$

This result is a direct consequence of Markoff's inequality; see [2, p. 94, Problem 4].

LEMMA 3. *For each polynomial $p(t) = c_0 + \sum_{i=1}^{n} c_i t^i$, $I = [-1, 1]$,*

$$\max\{| c_i |: i = 0, 1,..., n\} \leqslant \binom{n}{[(n+1)/2]} n! \| p \| \tag{5.2}$$

where $\| p \| = \max_I | p(t)|$.

Proof. Each $c_i = p^{(i)}(0)/i!$, $i = 0, 1,..., n$. Thus

$$| c_i | \leqslant \frac{1}{i!} \| p^{(i)} \| \leqslant \frac{n^2(n-1)^2 \cdots (n-i+1)^2}{i!} \| p \| \tag{5.3}$$

by the previous lemma. But

$$\binom{n}{[(n+1)/2]} \geqslant \binom{n}{i}, \qquad i = 0, 1,..., n,$$

and consequently, inequality (5.3) implies that

$$| c_i | \leqslant \binom{n}{[(n+1)/2]} n! \| p \|, \qquad i = 0, 1,..., n.$$

We note (5.2) is true for the constant polynomial $p(t) = c_0$ if we adopt the convention that $\binom{0}{0} = 1$.

THEOREM 3. *Suppose that the f of* IVP (1.1) *satisfies condition Q. For $\alpha \leqslant \bar{\alpha}$, let $p_k(t)$ and $p_k{}^*(t)$ be the* SAS *and* BAS *of degree k for the* IVP (1.1) *on the interval I_α. Then there exists a constant $\bar{N}$ such that for all $\alpha \leqslant \bar{\alpha}$,*

$$\| F[p_k] - F[p_k{}^*] \|_\alpha \leqslant \bar{N} \max_{i=0,1} \{\| p_k^{(i)} - p_k^{*(i)} \|_\alpha\}. \tag{5.4}$$

Proof. The remarks following the proof of Corollary 2 imply the sets $\{\| p_k \|_\alpha: \alpha \leqslant \bar{\alpha}\}$ and $\{\| \dot{p}_k \|_\alpha: \alpha \leqslant \bar{\alpha}\}$ are contained in an interval $[-N_1, N_1]$ independent of α. We now show that a similar result is valid for the sets

$$\{\| p_k{}^* \|_\alpha: \alpha \leqslant \bar{\alpha}\} \qquad \text{and} \qquad \{\| \dot{p}_k{}^* \|_\alpha: \alpha \leqslant \bar{\alpha}\}. \tag{5.5}$$

Define

$$\delta_k(t) = \ddot{p}_k^*(t) - F[p_k^*](t).$$

Then since p_k^* is a BAS for each k on I_α, it is clear that the set $\{\| \delta_k \|_\alpha : \alpha \leqslant \bar{\alpha}\}$ is bounded.

But

$$\max_{I_\alpha} | \delta_k(t)| = \max_{[-1,1]} | \delta_k(\alpha t)|.$$

Thus condition Q implies that the set

$$\{\delta_k(\alpha t) : \alpha \leqslant \bar{\alpha}\} \tag{5.6}$$

is a uniformly bounded family of polynomials defined on the interval $[-1, 1]$. If $p_k^*(t) = c_0 + c_1 t + a_2^* t^2 + \cdots + a_k^* t^k$, then

$$\begin{aligned}\delta_k(\alpha t) = {} & 2a_2^* + 6a_3^*\alpha t + \cdots + k(k-1)\, a_k^*\alpha^{k-2}t^{k-2} \\ & - f(\alpha t, c_0 + c_1\alpha t + \cdots + a_k^*\alpha^k t^k, c_1 + 2a_2^*\alpha t + \cdots + k a_k^*\alpha^{k-1}t^{k-1}).\end{aligned}$$

We note since each BAS depends on the interval I_α, each a_i^* may depend on α.

The constant term of $\delta_k(\alpha t)$ is $2a_2^* - f(0, c_0, c_1)$ and is bounded independent of α. Also (5.6) implies the coefficients of powers of t are uniformly bounded in α. Now suppose that $| a_2^* |, | a_3^* | \alpha, \ldots, | a_m^* | \alpha^{m-2}$ are all uniformly bounded in α. A careful examination of condition Q implies the coefficient of t^{m-1} involves only $(m+1)\, m a_{m+1}^* \alpha^{m-1}$ and coefficients already assumed to be bounded. Thus $a_{m+1}^*\alpha^{m-1}$ must be uniformly bounded in α. Induction thus implies the set $\{| a_i^* | \alpha^{i-2}\}_{i=2}^k$ is bounded independent of α. But

$$\begin{aligned}\| \ddot{p}_k^* \|_\alpha &= \max_{[-1,1]} | \ddot{p}_k^*(\alpha t)| \\ &\leqslant 2 | a_2^* | + 6 | a_3^* | \alpha + \cdots + k(k-1) | a_k^* | \alpha^{k-2}.\end{aligned}$$

Consequently $\{\| \ddot{p}_k^* \|_\alpha : \alpha \leqslant \bar{\alpha}\}$ is bounded, and this implies the sets (5.5) are also bounded. Without loss of generality, we can assume that these sets are also contained in the interval $[-N_1, N_1]$.

The mean value theorem and condition Q imply that if $(t, x, \bar{x})$ and $(s, y, \bar{y})$ are points of the set $S_{\bar{\alpha}} = I_{\bar{\alpha}} \times [-N_1, N_1]^2$, then

$$| f(t, x, \bar{x}) - f(s, y, \bar{y})| \leqslant \bar{N} \max\{| x - y |, | \bar{x} - \bar{y} |\}.$$

Since the above establishes that

$$(t, p_k(t), \dot{p}_k(t)) \qquad \text{and} \qquad (t, p_k^*(t), \dot{p}_k^*(t))$$

are in $S_{\bar{\alpha}}$ for every $\alpha \leqslant \bar{\alpha}$, the proof is complete.

Before proceeding to the last two theorems of this section, we state a fundamental theorem of approximation theory and a lemma related to this theorem. The lemma is due to Cline [3].

STRONG UNICITY THEOREM *Let* $G = \mathrm{span}\{1, t, \ldots, t^{n-1}\}$, $I_\alpha = [-\alpha, \alpha]$, *and let* P_0 *be the best approximation from* G *to a given continuous function* f. *Then there exists a constant* γ *depending on* f *such that for any element* P *of* G,

$$\| f - P \|_\alpha \geqslant \| f - P_0 \|_\alpha + \gamma \| P_0 - P \|_\alpha .$$

A more general statement and proof of this theorem can be found in [2, p. 80].

LEMMA 4 (Cline). *Let* $G = \mathrm{span}\{1, t, \ldots, t^{n-1}\}$, $I_\alpha = [-\alpha, \alpha]$ *and suppose that* $E = \{t_j\}_{j=1}^{n+1}$ *is an extremal set for* $f - P_0$, *where* $f \in C[-\alpha, \alpha]$ *and* P_0 *is the best approximation from* G *to* f. *For* $i = 1, 2, \ldots, n + 1$, *define* $q_i \in G$ *by* $q_i(t_j) = \mathrm{sgn}[f(t_j) - P_0(t_j)]$, $j = 1, \ldots, n + 1$, $j \neq i$. *Then the* γ *of the strong unicity theorem may be chosen to be*

$$\gamma = [\max_{1 \leqslant i \leqslant n+1} \| q_i \|_\alpha]^{-1}. \tag{5.7}$$

We note the polynomials $\{q_1, \ldots, q_{n+1}\}$ depend on I_α and due to the form of (5.7), it is assumed that $f \notin G$.

THEOREM 4. *Suppose the* f *of* IVP (1.1) *satisfies condition* Q. *For* $\alpha \leqslant \bar{\alpha}$, *let* $p_k(t)$ *and* $p_k^*(t)$ *be the* SAS *and* BAS *of degree* k, *respectively, for the* IVP (1.1) *on the interval* I_α. *Then there exists a* $\gamma > 0$ *such that for every* $\alpha \leqslant \bar{\alpha}$,

$$\| F[p_k] - F[p_k^*] \|_\alpha \geqslant \gamma \| \ddot{p}_k - \ddot{p}_k^* \|_\alpha . \tag{5.8}$$

Proof. By the Strong Unicity Theorem, Corollary 3 and the fact that p_k^* is a BAS, there is a positive number γ_α, possibly depending on α, such that

$$\begin{aligned} \gamma_\alpha \| \ddot{p}_k - \ddot{p}_k^* \|_\alpha &\leqslant \| \ddot{p}_k^* - F[p_k] \|_\alpha - \| \ddot{p}_k - F[p_k] \|_\alpha \\ &\leqslant \| \ddot{p}_k^* - F[p_k] \|_\alpha - \| \ddot{p}_k^* - F[p_k^*] \|_\alpha . \end{aligned}$$

Thus

$$\gamma_\alpha \| \ddot{p}_k - \ddot{p}_k^* \|_\alpha \leqslant \| F[p_k] - F[p_k^*] \|_\alpha . \tag{5.9}$$

Now (5.7) may be used to establish that there is a γ independent of α so that (5.9) holds for all $\alpha \leqslant \bar{\alpha}$. In fact, Lemma 4 implies we may choose

$$\gamma_\alpha = [\max_{1 \leqslant i \leqslant k} \| q_i \|_\alpha]^{-1}, \tag{5.10}$$

where each $q_i(t)$ is the polynomial of degree $k - 2$ or less that solves the interpolation problem

$$\begin{aligned} q_i(t_j) &= \operatorname{sgn}\{F[p_k](t_j) - \ddot{p}_k(t_j)\} \\ &= -\operatorname{sgn}[e_k(t_j)], \qquad j = 1, 2, \dots, k, \end{aligned}$$

$i \neq j$; where again $\{t_1, \dots, t_k\}$ is an extremal set for $e_k(t)$. Thus each $q_i(t)$ interpolates the function $-e_k(t)/\| e_k \|_\alpha$ at the points t_j, $j = 1, 2, \dots, k$; $j \neq i$. A classical theorem of approximation theory (see [2, p. 60]) establishes that

$$\left\| q_i + \frac{e_k}{\| e_k \|_\alpha} \right\|_\alpha \leqslant \frac{\| e_k^{(k-1)} \|_\alpha}{(k-1)! \, \| e_k \|_\alpha} \| W_i \|_\alpha ,$$

where

$$W_i(t) = \prod_{\substack{j=1 \\ i \neq j}}^{k} (t - t_j).$$

Condition Q implies $e_k(t)$ is a nonzero polynomial of degree N_k or less, so that Lemma 2 implies

$$\left\| q_i + \frac{e_k}{\| e_k \|_\alpha} \right\|_\alpha \leqslant \frac{N_k{}^2(N_k - 1)^2 \cdots (N_k - k + 2)^2 \, \| e_k \|_\alpha \, \| W_i \|_\alpha}{(k-1)! \, \| e_k \|_\alpha \, \alpha^{k-1}} .$$

Therefore

$$\left\| q_i + \frac{e_k}{\| e_k \|^\alpha} \right\|_\alpha \leqslant \frac{N_k{}^2(N_k - 1)^2 \cdots (N_k - k + 2)^2 \, 2^{k-1}}{(k-1)!} . \tag{5.11}$$

Let M_k equal the right-hand side of (5.11). Then (5.11) implies

$$\| q_i \|_\alpha \leqslant M_k + 1,$$

and consequently for all $\alpha \leqslant \bar{\alpha}$,

$$[\max_{1 \leqslant i \leqslant k} \| q_i \|_\alpha]^{-1} \geqslant (M_k + 1)^{-1}.$$

Thus from (5.9) and (5.10),

$$\gamma = (M_k + 1)^{-1} \tag{5.12}$$

is such that for all $\alpha \leqslant \bar{\alpha}$,

$$\| F[p_k] - F[p_k{}^*] \|_\alpha \geqslant \gamma \, \| \ddot{p}_k - \ddot{p}_k{}^* \|_\alpha .$$

The final theorem of this section partially answers question (b).

THEOREM 5. *Suppose the f of* IVP (1.1) *satisfies condition Q. For $\alpha \leqslant \bar{\alpha}$, let $p_k(t)$ and $p_k^*(t)$ be a* SAS *and* BAS *of degree $k \geqslant 2$ for the* IVP (1.1) *on the interval I_α. Then there exists an $\alpha^* \leqslant \bar{\alpha}$ such that on I_{α^*}, the* SAS *of degree k is unique and is the unique* BAS *of degree k.*

Proof. Suppose by way of contradiction that for every $\alpha^* \leqslant \bar{\alpha}$ there is an $\alpha \leqslant \alpha^*$ such that if p_k^* and p_k are the BAS and SAS of degree k on I_α, then $\ddot{p}_k^* \neq \ddot{p}_k$. From the definition of BAS it must be that

$$\| \ddot{p}_k^* - F[p_k^*] \|_\alpha \leqslant \| \ddot{p}_k - F[p_k] \|_\alpha .$$

Combining the results of Theorems 3 and 4,

$$\gamma \| \ddot{p}_k - \ddot{p}_k^* \|_\alpha \leqslant \bar{N} \max_{i=0,1} \{ \| p_k^{(i)} - p_k^{*(i)} \|_\alpha \}, \tag{5.13}$$

where γ and $\bar{N}$ are independent of α.

Let

$$p_k(t) = c_0 + c_1 t + a_2 t^2 + \cdots + a_k t^k$$

and

$$p_k^*(t) = c_0 + c_1 t + a_2^* t^2 + \cdots + a_k^* t^k.$$

Then for $i = 0, 1$,

$$\frac{\| p_k^{(i)} - p_k^{*(i)} \|_\alpha}{\| \ddot{p}_k - \ddot{p}_k^* \|_\alpha} = \frac{\left(\begin{aligned} &\| (i+1)(a_2 - a_2^*)\, t^{2-i} + (2i+1)(a_3 - a_3^*)\, t^{3-i} + \cdots \\ &\qquad + [(k-1)i + 1](a_k - a_k^*)\, t^{k-i} \|_\alpha \end{aligned} \right)}{\| 2(a_2 - a_2^*) + 6(a_3 - a_3^*)t + \cdots + k(k-1)(a_k - a_k^*)\, t^{k-2} \|_\alpha} .$$

Again we note a_j and a_j^*, $j = 2, \ldots, k$, depend on α. Then

$$\frac{\| p_k^{(i)} - p_k^{*(i)} \|_\alpha}{\| \ddot{p}_k - \ddot{p}_k^* \|_\alpha} \leqslant \frac{\left(\begin{aligned} &\alpha^{2-i}(k-1) \max\{ (i+1) \mid a_2 - a_2^* \mid, \\ &\qquad (2i+1) \mid a_3 - a_3^* \mid \alpha, \ldots, [(k-1)i + 1] \mid a_k - a_k^* \mid \alpha^{k-2} \} \end{aligned} \right)}{\left(\begin{aligned} &\max_{[-1,1]} \mid 2(a_2 - a_2^*) + 6(a_3 - a_3^*)\, \alpha t + \cdots \\ &\qquad + k(k-1)(a_k - a_k^*)\, \alpha^{k-2} t^{k-2} \mid \end{aligned} \right)} .$$

Now Lemma 3 implies

$$\frac{\| p_k^{(i)} - p_k^{*(i)} \|_\alpha}{\| \ddot{p}_k - \ddot{p}_k^* \|_\alpha} \leqslant \alpha^{2-i} \binom{k-2}{[(k-1)/2]} (k-1)!\, \frac{\mu_{ki}}{\eta_k} ,$$

where

$$\mu_{ki} = \max\{(i+1) \mid a_2 - a_2^* \mid, (2i+1) \mid a_3 - a_3^* \mid \alpha, \ldots, [(k-1)i+1] \mid a_k - a_k^* \mid \alpha^{k-2}\}$$

and

$$\eta_k = \max\{2 \mid a_2 - a_2^* \mid, 6 \mid a_3 - a_3^* \mid \alpha, \ldots, k(k-1) \mid a_k - a_k^* \mid \alpha^{k-2}\}.$$

Since $\max\{\mu_{k0}/\eta_k, \mu_{k1}/\eta_k\} \leqslant 1$, it follows that

$$\frac{\| p_k^{(i)} - p_k^{*(i)} \|_\alpha}{\| \ddot{p}_k - \ddot{p}_k^* \|_\alpha} \leqslant \alpha^{2-i} \binom{k-2}{[(k-1)/2]} (k-1)! . \tag{5.14}$$

Thus for every $\alpha \leqslant \bar{\alpha}$, (5.14) implies

$$\| p_k^{(i)} - p_k^{*(i)} \|_\alpha \leqslant \alpha^{2-i} \mathscr{E}_k \| \ddot{p}_k - \ddot{p}_k^* \|_\alpha , \qquad i = 0, 1,$$

where

$$\mathscr{E}_k = \binom{k-2}{[(k-1)/2]} (k-1)! \tag{5.15}$$

Hence from (5.13),

$$\gamma \leqslant \bar{N} \mathscr{E}_k \max_{i=0,1} \{\alpha^{2-i}\}.$$

But for α small enough, this is a contradiction. Therefore, for fixed k, there exists an $\alpha^* \leqslant \bar{\alpha}$ such that $\ddot{p}_k(t) \equiv \ddot{p}_k^*(t)$, and consequently the SAS and BAS of degree k are equal on I_{α^*}.

We note in concluding the proof of this theorem that if α^* is sufficiently small, the above analysis actually implies the SAS and BAS of degree k are equal and unique on every I_α, $\alpha \leqslant \alpha^*$.

Theorem 5 is deficient in that the interval over which the SAS and BAS are equal depends on k. However, the $I_{\bar{\alpha}}$ over which the SAS of Corollary 1 exists is independent of k.

6. Error Analysis and Examples

Suppose $y_k(t)$ is a solution to the IVP

$$\ddot{x}(t) = f(t, x(t), \dot{x}(t)) + v_k(t)$$
$$x(0) = c_0, \qquad \dot{x}(0) = c_1$$

on the interval $I_\alpha = [-\alpha, \alpha]$, where $f(t, x, \bar{x})$ and its first partial derivatives with respect to the last two variables are continuous on $I_\alpha \times R^2$. Assume for each k that $v_k \in C[-\alpha, \alpha]$, and that the sequence $\{\| y_k \|\}_{k=2}^{\infty}$, is bounded. If $y(t)$ is the solution to (1.1) on I_α, then

$$\max_{i=0,1} \{\| y_k^{(i)} - y^{(i)} \|_\alpha\} \leqslant N^* \| v_k \|_\alpha , \tag{6.1}$$

where N^* is a nonnegative constant not depending on k. The proof of this inequality involves a standard Gronwall inequality argument.

Using Eq. (4.7) and inequality (6.1) the error estimate

$$\max_{i=0,1} \{\| p_k^{(i)} - y^{(i)} \|_\alpha\} \leqslant N^* \| e_k \|_\alpha , \qquad \alpha \leqslant \bar{\alpha}, \tag{6.2}$$

is obtained, where $\{p_k\}_{k=2}^{\infty}$ is a sequence of SAS's on I_α. Also $\| e_k \|_\alpha$ in (6.2) is the maximum deviation that arises by employing the Remes algorithm to obtain the SAS of degree k, see Corollary 3. Thus except for the fixed constant N^*, (6.2) provides an error estimate. Inequality (6.2) actually does not rely on $\alpha \leqslant \bar{\alpha}$, and hence may be used globally ($\alpha > \bar{\alpha}$) in the computations if the data suggests that $\| p_k \|_\alpha$ is bounded for large k.

We conclude this paper with several numerical examples. A Fortran IV program using double precision arithmetic for the SIGMA 7 computer is utilized in the calculations.

EXAMPLE 3.

$$\ddot{x}(t) - x(t)\,\dot{x}(t) = -\tfrac{1}{2}(\sin 2t + 2 \sin t), \qquad I = [-1, 1]$$
$$x(0) = 0, \qquad \dot{x}(0) = 1.$$

If $k = 5$ approximation is from $\mathbf{P}_5 = \{t + a_2t^2 + a_3t^3 + a_4t^4 + a_5t^5\}$. Then $p_{5n}(t) = t + a_{2n}t^2 + a_{3n}t^3 + a_{4n}t^4 + a_{5n}t^5$ is the nth iterate of algorithm (3.2); let $A_{5n} = (a_{2n}, a_{3n}, a_{4n}, a_{5n})$. Initially, $A_{50} = (0, 0, 0, 0)$. Further iterations yield

$$A_{51} = (0, -0.16026, 0, 0.03365),$$
$$A_{52} = (0, -0.17187, 0, 0.01647),$$
$$A_{53} = (0, -0.16833, 0, 0.00953),$$
$$\cdots$$
$$A_{58} = (0, -0.16621, 0, 0.00783),$$

and $A_{58} = A_{59} = A_{510}$. Thus the SAS of degree 5 is

$$p_5(t) = t - 0.16621t^3 + 0.00783t^5.$$

The solution to this IVP is $y(t) = \sin t$, and $\| y - p_5 \| = 0.00014$. The predicted error is $\| e_5 \| = 0.00052$, where the terminology predicted error refers to the error estimate (6.2) with the omission of the factor N^*. For this example Corollary 1 guarantees the existence of a SAS of degree k if $\alpha \leqslant \bar{\alpha} = 0.11$ but algorithm (3.2) actually converges for $\alpha \leqslant 1$. However, convergence is faster for smaller values of α.

EXAMPLE 4.

$$\ddot{x}(t) - 2t\dot{x}(t) - 2x(t) = 0, \qquad I = [-0.7, 0.7],$$
$$x(0) = 1, \qquad \dot{x}(0) = 0.$$

Since this is a linear IVP, a BAS actually can be computed [5]. The BAS of degree four is

$$p_4{}^*(t) = 1 + 0.88746t^2 + 0.74996t^4.$$

Algorithm (3.2) finally yields $p_4 = p_{4,13} = p_4{}^*$. The solution to the IVP is $y(t) = e^{t^2}$, and $\| y - p_4 \| = 0.01739$.

EXAMPLE 5.

$$\ddot{x}(t) + 2(t - 2)\,\dot{x}(t)[x(t)]^2 = 0, \qquad I = [-0.7, 0.7],$$
$$x(0) = \tfrac{1}{2}, \qquad \dot{x}(0) = \tfrac{1}{4}.$$

The solution is $y(t) = 1/(2 - t)$. If the algorithm is initiated at $k = 5$, 12 iterations yield the SAS of degree five,

$$p_5(t) = 0.5 + 0.25t + 0.11988t^2 + 0.05709t^3 + 0.04250t^4 + 0.02335t^5,$$

and $\| y - p_5 \| = 0.00177$. In this case the predicted error is $\| e_5 \| = 0.01097$.

The $f(t, x, \bar{x})$ of Examples 4–5 satisfy condition Q. The last example of this paper is included to suggest that the algorithm may be effective in more general circumstances.

EXAMPLE 6.

$$\ddot{x}(t) + \frac{[x(t)]^2}{\dot{x}(t)} = -\sin t(1 - \tan t), \qquad I = [-1, 1],$$
$$x(0) = 0, \qquad \dot{x}(0) = 1.$$

The solution is $y(t) = \sin t$, and approximation is from $\mathbf{P}_5$. Twelve iterations of the algorithm yield the SAS of degree five;

$$p_5(t) = t + 0.00001t^2 - 0.166249t^3 - 0.00001t^4 + 0.00782t^5.$$

The predicted error is $\| e_5 \|_\alpha = 0.00049$, and the actual error is $\| y - p_5 \|_\alpha = 0.00010$.

7. Conclusions

Although the nonlinear best approximation problem (3.1) has been extensively studied in the literature, few attempts toward computation of best simultaneous approximation solutions have been made, basically because the approximation problem is nonlinear. The simultaneous approximation substitute (3.2) was introduced in [10], and Sections 2–5 of this paper provide a theory that closely relates BAS and SAS. Finally, Section 6 demonstrates for k sufficiently large that a SAS of degree k is often a good approximation to the solution of IVP (1.1), even when theoretical requirements are not satisfied.

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